Quantum propagator for some classes of three-dimensional three-body systems

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Abstract

In this work we solve exactly a class of three-body propagators for the most general quadratic interactions in the coordinates, for arbitrary masses and couplings. This is done both for the constant as the time-dependent couplings and masses, by using the Feynman path integral formalism. Finally the energy spectrum and the eigenfunctions are recovered from the propagators.

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1 Introduction

In a very recent work, A. Chouchaoui [1] calculated the propagator for the problem of three identical particles in one dimension. This was done by working in the Feynman path integral formalism [2][3], so expanding the class of three-body systems solved through this formalism as, for instance, those discussed by Khandekhar and collaborators [4], Govaerts [5], and others.

On the other hand, the interest in solving problems involving time-dependent systems has attracted the attention of physicists since a long time. This happens due to its applicability for the understanding of many problems in quantum optics, quantum chemistry and others areas of physics [6]-[17]. In particular we can cite the case of the electromagnetic field intensities in a Fabry-Pérot cavity [6]. In fact this kind of problem still represents a line of investigation which attract the interest of physicists [16]-[20].

Here we intend to expand the class of exactly solvable path integral problems, by including the case of a general three-body quadratic interaction in the coodinates, both in the case of constant as in the case of time-dependent couplings. This is going to be done through a suitable combination of the Jacobi coordinates with a further decoupling one.

2 A general exactly solvable quadratic interactions for three-body with constant couplings

The model which we are going to treat is represented by the following Lagrangian

$$\begin{aligned}\mathcal{L} &= \sum_{j=1}^{3}\frac{m_j}{2}\left(\frac{d\vec{r}_j}{dt}\right)^2-\frac{1}{2}\Big[K_{21}\,(\vec{r}_2-\vec{r}_1)^2+K_{31}\,(\vec{r}_3-\vec{r}_1)^2+K_{32}\,(\vec{r}_3-\vec{r}_2)^2+\\ &\quad+\sigma_1(\vec{r}_2-\vec{r}_1)\cdot(\vec{r}_3-\vec{r}_1)+\sigma_2(\vec{r}_2-\vec{r}_1)\cdot(\vec{r}_3-\vec{r}_2)+\\ &\quad+\sigma_3(\vec{r}_3-\vec{r}_1)\cdot(\vec{r}_3-\vec{r}_2)+\vec{g}_1\cdot(\vec{r}_2-\vec{r}_1)+\vec{g}_2\cdot(\vec{r}_3-\vec{r}_1)+\vec{g}_3\cdot(\vec{r}_3-\vec{r}_2)\Big].\end{aligned} \tag{1}$$

As far as we know, this problem was considered only for particular cases, in general without the crossed terms proportional to the σ coupling constants [21], and most part of time for one-dimensional identical particles under

isotropic harmonic interactions [1],[4], [5]. As a matter of fact, we should say that, in fact the parameters $\vec{g}_1$, $\vec{g}_2$ and $\vec{g}_3$ can really be time-dependent as we are going to see below in the text.

For the first step, in order to reach our goal, we perform a change of coordinates system going to the so called Jacobi coordinates, which are characterized in this case by the following set of transformations [21]:

$$\begin{pmatrix} \vec{X}_1 \\ \vec{X}_2 \\ \vec{X}_3 \end{pmatrix} = \begin{pmatrix} -a & a & 0 \\ -\frac{b\,m_1}{m_{12}} & -\frac{b\,m_2}{m_{12}} & b \\ \frac{m_1}{M} & \frac{m_2}{M} & \frac{m_3}{M} \end{pmatrix} \begin{pmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{pmatrix}, \tag{2}$$

where a, b are arbitrary constants, used in order to put the Lagrangian into a more convenient form, and $M \equiv m_1 + m_2 + m_3$ is the total mass of the system. Note that for the sake of path-integral evaluation, we must compute the Jacobian of the transformation due to the impact of the transformation over the measure of the path-integral, and in for this transformation it is given simply by $(ab)^3$. As observed in [21], this transformation is not sufficient to decouple the system, which will looks like

$$\begin{aligned} \mathcal{L} \;=\; & \sum_{j=1}^{3} \frac{M_j}{2} \left(\frac{d\vec{X}_j}{dt} \right)^2 - \frac{1}{2} \Big[M_1 \omega_1^2 \,\vec{X}_1^2 + M_2 \omega_2^2 \,\vec{X}_2^2 + \lambda \;\; \vec{X}_1 \cdot \vec{X}_2 + \\ & + \vec{f}_1\,(t) \cdot \vec{X}_1 + \vec{f}_2\,(t) \cdot \vec{X}_2 \Big] , \end{aligned} \tag{3}$$

with the resulting masses and coupling constants given by

$$\begin{aligned} M_1 \;&\equiv\; \frac{m_1 m_2}{a^2 m_{12}};\;\; M_2 \equiv \frac{m_3\, m_{12}}{b^2\, M};\;\; M_3 \equiv M; \\ \omega_1^2 \;&\equiv\; \frac{m_{12}}{m_1 m_2} \Bigg\{ K_{21} + \frac{1}{m_{12}^2} \left(m_2^2 K_{31} + m_1^2 K_{32} \right) + \\ &\quad + \frac{2}{m_{12}} \left[\sigma_1 m_2 - \sigma_2 m_1 - \sigma_3 \left(\frac{m_1 m_2}{m_{12}} \right) \right] \Bigg\} ; \\ \omega_2^2 \;&\equiv\; \frac{M}{m_3 m_{12}} \left(K_{31} + K_{32} + 2\,\sigma_3 \right); \\ \lambda \;&\equiv\; \frac{1}{a\,b} \left[\frac{\sigma_3}{m_{12}} \left(m_2 - m_1 \right) + \sigma_1 + \sigma_2 + \frac{1}{m_{12}} \left(m_2 K_{31} - m_1 K_{32} \right) \right]; \end{aligned} \tag{4}$$

$$\vec{f}_1 \equiv \frac{1}{a}\left(\vec{g}_1 + \frac{(m_2\vec{g}_2 - m_1\vec{g}_3)}{m_{12}}\right);\ \vec{f}_2 \equiv \frac{1}{b}\left(\vec{g}_2 + \vec{g}_3\right);$$

with $m_{12} \equiv m_1 + m_2$.

Note that the center of mass coordinate has decoupled entirely from the other ones, so that one can now worry only about the remaining coordinate variables. Now one can take two possible routes. The first, which is the one usually taken, sometimes with errors as observed in [21], by particularizing the problem through a constraint over the parameters: $\lambda \equiv 0$. Another possible and more general route, is the one we are going to choose here. In such route we perform a further transformation, as proposed in [22] and used also in [21], keeping in mind the need of decoupling the final system through the elimination of the crossed term. This is reached by a simple additional dilation and rotation transformation [22], which in the present situation is defined as

$$\begin{pmatrix} \vec{X}_1 \\ \vec{X}_2 \end{pmatrix} = \begin{pmatrix} \left(\sqrt{\frac{m}{M_1}}\right)\cos\phi & \left(\sqrt{\frac{m}{M_1}}\right)\sin\phi \\ -\left(\sqrt{\frac{m}{M_2}}\right)\sin\phi & \left(\sqrt{\frac{m}{M_2}}\right)\cos\phi \end{pmatrix}\begin{pmatrix} \vec{Y}_1 \\ \vec{Y}_2 \end{pmatrix},\ \vec{X}_3 \equiv \vec{Y}_3, \tag{5}$$

where m is an arbitrary parameter with dimensions of mass, and ϕ is the rotation angle which are going to be fixed in order to disentangle the system.

After using these transformations, one achieves the following Lagrangian for the last two variables

$$\begin{aligned} \mathcal{L} &= \frac{M}{2}\left(\frac{d\vec{Y}_3}{dt}\right)^2 + \sum_{j=1}^{2}\frac{m}{2}\left(\frac{d\vec{Y}_j}{dt}\right)^2 - \frac{1}{2}\left[\alpha\,\vec{Y}_1^2 + \beta\,\vec{Y}_2^2 + \gamma\,\vec{Y}_1\cdot\vec{Y}_2 + \right. \\ &\quad \left. + \vec{F}_1(t)\cdot\vec{Y}_1 + \vec{F}_2(t)\cdot\vec{Y}_2\right], \end{aligned} \tag{6}$$

with the transformed constant couplings being defined as

$$\begin{aligned} \alpha &\equiv m\,\omega_1^2\cos\phi^2 + m\,\omega_2^2\sin\phi^2 - \frac{\lambda\,m}{\sqrt{M_1M_2}}\sin(2\phi); \\ \beta &\equiv m\,\omega_1^2\sin\phi^2 + m\,\omega_2^2\cos\phi^2 + \frac{\lambda\,m}{\sqrt{M_1M_2}}\sin(2\phi); \\ \gamma &\equiv m\left(\omega_1^2 - \omega_2^2\right)\sin(2\phi) + \frac{2\,\lambda\,m}{\sqrt{M_1M_2}}\cos(2\phi); \end{aligned} \tag{7}$$

and

$$\vec{F}_1 \equiv \sqrt{\frac{m}{M_1}}\,\vec{f}_1\,\cos\phi - \sqrt{\frac{m}{M_2}}\,\vec{f}_2\,\sin\phi;$$
$$\vec{F}_2 \equiv \sqrt{\frac{m}{M_1}}\,\vec{f}_1\,\sin\phi + \sqrt{\frac{m}{M_2}}\,\vec{f}_2\,\cos\phi. \tag{8}$$

Furthermore, due to the above transformation we get a change of the path-integral measure given by a cubic power of the Jacobian of the transformation ($J^3 = \left(\frac{\sqrt{M_1 M_2}}{m}\right)^3$). Now we are in conditions to eliminate the crossed term in these last coordinates, and this is done by imposing that the angle ϕ should obeys:

$$\tan(\phi) = \frac{2\,\lambda}{\sqrt{M_1 M_2}}\left(\omega_2^2 - \omega_1^2\right). \tag{9}$$

Solving the above equation, one can easily to note that two solutions appear, but they just interchange the role of the new vectors $\vec{Y}_1$ and $\vec{Y}_2$ in the Lagrangian, which lead us to conclude that both conduce to the same physical consequences, in such a way that we only need to work with one of them. We will use the following solution

$$\cos\phi = \frac{1}{2}\left(1 + \frac{\sqrt{M_1 M_2\,(\omega_2^2 - \omega_1^2)^2}}{\sqrt{4\,\lambda^2 + M_1 M_2\,(\omega_2^2 - \omega_1^2)^2}}\right). \tag{10}$$

Using the above solution, the Lagrangian will be finally set decoupled and given by

$$\mathcal{L} = \frac{m}{2}\sum_{j=1}^{2}\left[\left(\frac{d\vec{Y}_j}{dt}\right)^2 - \Omega_j^2\,\vec{Y}_j^2 + \frac{2}{m}\vec{F}_j\,(t)\cdot\vec{Y}_j\right], \tag{11}$$

where the final decoupled frequencies can be written as

$$\Omega_1^2 \equiv \frac{1}{2}\left\{\omega_1^2 + \omega_2^2 - \left[\left(\omega_2^2 - \omega_1^2\right)^2 + \frac{4\,\lambda}{M_1 M_2}\right]^{\frac{1}{2}}\right\},$$
$$\Omega_2^2 \equiv \frac{1}{2}\left\{\omega_1^2 + \omega_2^2 + \left[\left(\omega_2^2 - \omega_1^2\right)^2 + \frac{4\,\lambda}{M_1 M_2}\right]^{\frac{1}{2}}\right\}, \tag{12}$$

and the forces

$$\begin{aligned}
\vec{F}_1\,(t) &\equiv \sqrt{\frac{m}{2\,M_1}\,(1+R)}\,\vec{f}_1\,(t)\;-\;\sqrt{\frac{m}{2\,M_2}\,(1-R)}\,\vec{f}_2\,(t)\;, \\
\vec{F}_2\,(t) &\equiv \sqrt{\frac{m}{2\,M_1}\,(1-R)}\,\vec{f}_1\,(t)\;+\;\sqrt{\frac{m}{2\,M_2}\,(1+R)}\,\vec{f}_2\,(t)\;,
\end{aligned} \tag{13}$$

where we defined that

$$R \equiv \frac{\sqrt{M_1 M_2\left(\omega_2^2-\omega_1^2\right)^2}}{\sqrt{4\,\lambda^2 + M_1 M_2\left(\omega_2^2-\omega_1^2\right)^2}}. \tag{14}$$

Finally we are left with the task of solving the corresponding Feynman propagator for a system of uncoupled forced harmonic oscillators. For this we can use the well known solutions of this system [2].

Remembering the fact that the final Lagrangian is a direct sum of independent ones, the corresponding propagator must be just the product of three independent propagators in terms of the final variables, so we get

$$\begin{aligned}
K\left(\vec{Y}_1^{\,\prime\prime},\vec{Y}_1^{\,\prime};\vec{Y}_2^{\,\prime\prime},\vec{Y}_2^{\,\prime};\vec{Y}_3^{\,\prime\prime},\vec{Y}_3^{\,\prime};\tau\right) &= \left(a\,b\,\frac{\sqrt{M_1 M_2}}{m}\right)^3 K_1\left(\vec{Y}_1^{\,\prime\prime},\vec{Y}_1^{\,\prime};\tau\right)\times \\
&\quad \times K_2\left(\vec{Y}_2^{\,\prime\prime},\vec{Y}_2^{\,\prime};\tau\right)\,K_3\left(\vec{Y}_3^{\,\prime\prime},\vec{Y}_3^{\,\prime};\tau\right),
\end{aligned} \tag{15}$$

where the prime and the double prime denotes that a function is being evaluated at the initial or at the final instant of the time interval respectively. Besides $\vec{\tau} \equiv t_b - t_a$ is the time interval between these instants. The first propagator K_3 is that of a free three-dimensional particle with the total mass of the system of three particles, and can be easily obtained [2], as

$$K_3\left(\vec{Y}_3^{\,\prime\prime},\vec{Y}_3^{\,\prime};\tau\right) = \left(\frac{M}{2\,\pi\,i\,\tau}\right)^{\frac{3}{2}} \exp\left\{\frac{i\,M}{2\,\hbar\,\tau}\left(\vec{X}_3^{\,\prime\prime}-\vec{X}_3^{\,\prime}\right)^2\right\}, \tag{16}$$

once $\vec{Y}_3 = \vec{X}_3$. The other two have the same form of a three-dimensional driven harmonic oscillator [2]:

$$\begin{aligned}
K_j\left(\vec{Y}_j^{\,\prime\prime},\vec{Y}_j^{\,\prime};\tau\right) &= \left[\frac{m\,\Omega_j}{2\,\pi\,\hbar\;i\;\sin\left(\Omega_j\,\tau\right)}\right]^{\frac{3}{2}} \exp\left\{\frac{i\;m\,\Omega_j}{2\;\hbar\;\sin\left(\Omega_j\,\tau\right)}\left[\left(\left(\vec{Y}_j^{\,\prime\prime}\right)^2 + \right.\right.\right. \\
&\quad \left.\left.\left. + \left(\vec{Y}_j^{\,\prime}\right)^2\right)\cos\left(\Omega_j\,\tau\right) - 2\,\vec{Y}_j^{\,\prime\prime}\cdot\vec{Y}_j^{\,\prime} - G_j\left(\tau\right)\right]\right\}
\end{aligned} \tag{17}$$

where

$$
\begin{aligned}
G_j(\tau) \;\equiv\;& \left(\frac{2}{m\,\Omega_j}\right)\left\{\int_{t_b}^{t_a} dt\,\left[\left(\vec{Y}_j'' \cdot \vec{F}_j(t)\right)\,\sin\left[\Omega_j\,(t-t_a)\right]+\right.\right.\\
&+\left.\left.\left(\vec{Y}_j' \cdot\, \vec{F}_j(t)\right)\,\sin\left[\Omega_j\,(t_b-t)\right]\right]\right\}+\\
&-\left(\frac{2}{m^2\,\Omega_j^2}\right)\int_{t_a}^{t_b} dt' \int_{t_a}^{t} dt\,\left(\vec{F}_j(t)\,\cdot\,\vec{F}_j(t')\right)\,\times\\
&\times\sin\left[\Omega_j\,(t_b-t)\right]\sin\left[\Omega_j\,\left(t'-t_a\right)\right],
\end{aligned}
\tag{18}
$$

and $j=1,\,2$. Now, by using the above expression, one can finally write the final expression of the quantum propagator of the three-body system under examination, simply multiplying the three resulting propagators, and further taking back the original physical variables, through

$$
\begin{pmatrix} \vec{Y}_1 \\ \vec{Y}_2 \end{pmatrix} = \begin{pmatrix} \left(\sqrt{\frac{M_1}{m}}\right)\cos\phi & -\left(\sqrt{\frac{M_2}{m}}\right)\sin\phi \\ \left(\sqrt{\frac{M_1}{m}}\right)\sin\phi & \left(\sqrt{\frac{M_2}{m}}\right)\cos\phi \end{pmatrix} \begin{pmatrix} \vec{X}_1 \\ \vec{X}_2 \end{pmatrix}, \tag{19}
$$

and, also using the initial transformation (2). Here, however, we avoid to write the final expression for the sake of conciseness and, because at this point it is only a simple task of substituting and multiplying the decoupled propagators.

From the above expressions we can extract the corresponding wave functions and energies. For this we remember that the propagator can be obtained from the following spectral summation

$$
K\left(z'',z';\tau,0\right)=\sum_{n=0}^{\infty}\psi_n^*\left(z',t'\right)\,\psi_n\left(z'',t''\right). \tag{20}
$$

On the other hand, we can use the Mehler's formula [25],

$$
\frac{\exp\left[-\left(a^2+b^2-2\,a\,b\,c\right)/\left(1-c^2\right)\right]}{\sqrt{\left(1-c^2\right)}}=\exp\left[-\left(x^2+b^2\right)\right]\sum_{n=0}^{\infty}\frac{c^n}{n!}H_n(a)\,H_n(b)\,, \tag{21}
$$

in order to [22] recover the corresponding wave functions. After length but straightforward calculations one can finally obtain a wave function which consists of a three-dimensional free particle with continuous energy, and three

three-dimensional driven harmonic oscillators, with discrete eigenenergies. Once more, in order to be concise, we will write below only the quantized part of the wave function,

$$\Psi = \Pi_{a=1}^{3} \psi_{n_{1a} n_{2a}} \left(a_1, a_2\right), \tag{22}$$

with

$$\begin{aligned}
\psi_{n_{1a} n_{2a}} \left(a_1, a_2\right) \;\; = \;\; & \frac{1}{2^{-(n_{1a}+n_{2a})} \left(n_{1a}!\, n_{2a}!\right)^{\frac{1}{2}}} \left(\frac{M_1 M_2 \Omega_1 \Omega_2}{\pi^2 \hbar^2}\right)^{\frac{1}{4}} \exp\left(-i\, E_{n_{1a} n_{2a}}\, t\right) \\
& \times \exp\left\{ \left(\frac{1}{2\,\hbar}\right) \left[-\Omega_1 \left(\sqrt{M_1}\, C\, a_1 - \sqrt{M_2}\, S\; a_2 + \eta_{1a} \right)^2 \right] \right. \\
& -\Omega_2 \left(\sqrt{M_1}\, S\, a_1 - \sqrt{M_2}\, C\, a_2 + \eta_{2a} \right)^2 + \\
& -i\, \dot{\eta}_{1a} \left[\eta_{1a} - 2 \left(\sqrt{M_1}\, C\, a_1 - \sqrt{M_2}\, S\, a_2 \right) \right] + \\
& \left. -i\, \dot{\eta}_{2a} \left[\eta_{2a} - 2 \left(\sqrt{M_1}\, S\; a_1 - \sqrt{M_2}\, C\, a_2 \right) \right] \right\} \\
& \times \exp\left\{ -\frac{i}{2\hbar} \int^t d\lambda \left[\left(\frac{\eta_{1a}}{\sqrt{M_1}} \left(F_{1a}\, C - f_2\, S\right) \right) + \right. \right. \\
& \left. \left. \left(\frac{\eta_{2a}}{\sqrt{M_2}} \left(F_{2a}\, S + f_2\, C\right) \right) \right] \right\} \\
& \times H_{n_{1a}} \left[\sqrt{\frac{\Omega_1}{\hbar}} \left(\sqrt{M_1}\, C\, a_1 - \sqrt{M_2}\, S\, a_2 \right) \right] \\
& \times H_{n_{2a}} \left[\sqrt{\frac{\Omega_2}{\hbar}} \left(\sqrt{M_1}\, S\, a_1 + \sqrt{M_2}\, C\, a_2 \right) \right],
\end{aligned} \tag{23}$$

where $a = 1, 2, 3 = x, y, z$, $S \equiv \sin(\phi)$, $C \equiv \cos(\phi)$, $H_{n_{i\,a}}(\cdot)$ is the $n_{i\,a}$ th Hermite polynomial, and

$$\begin{aligned}
\eta_{ia}(t) \;\; \equiv \;\; & \left(\frac{1}{\sqrt{m}\,\sin\left(\Omega_i\, t\right)}\right) \left\{ \int_{t_a}^{t} d\xi\, F_{ia}(\xi)\, \sin\left(\Omega_i \left(\xi - t_a\right)\right) \sin\left(\Omega_i \left(t_{b+} - t\right)\right) \right. \\
& \left. + \int_{t}^{t_b} d\xi\, F_{ia}(\xi)\, \sin\left(\Omega_i \left(\xi - t_a\right)\right) \sin\left(\Omega_i \left(t_b - t\right)\right) \right\}.
\end{aligned} \tag{24}$$

Particularly in the case of non-driving forces ($\vec{g}_1 = \vec{g}_2 = \vec{g}_3 = 0$), the eigen-energies are given by

$$E = \sum_{a=1}^{3} E_{n_{1a}n_{2a}} = \sum_{a=1}^{3} \left[\left(n_{1a} + \frac{1}{2} \right) \hbar\, \Omega_1 + \left(n_{2a} + \frac{1}{2} \right) \hbar\, \Omega_2 \right], \tag{25}$$

from which we can observe that, beyond the usual degeneracy which happens when $\sum_{a=1}^{3} n_{1a} = N_1$, or $\sum_{a=1}^{3} n_{2a} = N_2$, with N_1 and N_2 being integer numbers. In the case where the final frequencies have a rational relation, further degeneracies will appear. As the transformed frequencies are functions of the original ones and of the couplings, one concludes that, when certain relations between the parameters hold, the system becomes more degenerate, signalizing the appearance of hidden symmetries.

3 A general exactly solvable quadratic interactions for three-body with time-dependent couplings

In this section, we extend our calculation in order to include the case of time-dependent couplings in the three-body system. In this case the number of works in this matter is even more scarce and, up to our knowledge, restrict to the one-dimensional case [4], [5]. In this new situation, as a consequence of the fact that if we try to do the same above transformations, additional terms would appear rendering the system unsolvable if the masses were time-dependent and we must keep the masses constant. Furthermore, we should restrict ourselves to treat the case of isotropic frequencies, and perform some new transformations because the decoupled system is still time-dependent [11]. In view of these arguments, we deal with a system characterized by the Lagrangian density (1) with arbitrary constant masses, perform the transformations (2) getting a Lagrangian density with the same form of (3) but now, one has time-dependent frequencies and couplings. However, in order to guarantee the exact solvability we must impose the following set of constraints

$$M_1 \equiv \frac{m_1 m_2}{a^2 m_{12}} \;=\; M_2 \equiv \frac{m_3\, m_{12}}{b^2\, M} \equiv \mu,\, \omega_1\,(t)^2 = \;\omega_2\,(t)^2\,, \tag{26}$$

which leads us to fix one of the arbitrary constants through the relation

$$\left(\frac{a}{b}\right) \equiv \sqrt{\frac{m_1 m_2 M}{m_3 m_{12}^2}}. \tag{27}$$

On the other hand, the constraint among the frequencies implies into a restriction over the time-dependency of one of the coupling parameters, given by

$$\begin{aligned} \omega^2(t) &\equiv \omega_1^2 \equiv \frac{m_{12}}{m_1 m_2}\left\{K_{21} + \frac{1}{m_{12}^2}\left(m_2^2 K_{31} + m_1^2 K_{32}\right) + \right. \\ &\left. +\frac{2}{m_{12}}\left[\sigma_1 m_2 - \sigma_2 m_1 - \sigma_3\left(\frac{m_1 m_2}{m_{12}}\right)\right]\right\} = \\ &= \omega_2^2 \equiv \frac{M}{m_3 m_{12}}\left(K_{31} + K_{32} + 2\,\sigma_3\right). \end{aligned} \tag{28}$$

Leading to the following Lagrangian

$$\begin{aligned} \mathcal{L} &= \frac{M}{2}\left(\frac{d\vec{X}_3}{dt}\right)^2 + \sum_{j=1}^{2}\frac{\mu}{2}\left(\frac{d\vec{X}_j}{dt}\right)^2 - \frac{1}{2}\left[\mu\,\omega(t)^2\left(\vec{X}_1^2 + \vec{X}_2^2\right) + \right. \\ &\left. + \lambda(t)\;\; \vec{X}_1 \cdot \vec{X}_2 + \vec{f}_1(t) \cdot \vec{X}_1 + \vec{f}_2(t) \cdot \vec{X}_2\right], \end{aligned} \tag{29}$$

Now, we can decouple the coordinates $\vec{X}_1$ and $\vec{X}_2$, by performing a $\pi/4$ rotation around the third vector like

$$\begin{pmatrix} \vec{X}_1 \\ \vec{X}_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \end{pmatrix}, \tag{30}$$

and this leads us to the following Lagrangian

$$\begin{aligned} \mathcal{L} &= \frac{M}{2}\left(\frac{d\vec{X}_3}{dt}\right)^2 + \sum_{j=1}^{2}\frac{\mu}{2}\left(\frac{d\vec{x}_j}{dt}\right)^2 - \frac{\mu}{2}\left[\Omega_1(t)^2\;\vec{x}_1^2 + \Omega_2(t)^2\,\vec{x}_2^2\right. \\ &\left. +\vec{\theta}_1(t) \cdot \vec{x}_1 + \vec{\theta}_2(t) \cdot \vec{x}_2\right], \end{aligned} \tag{31}$$

where the frequencies and linear force-type couplings of the decoupled three-dimensional oscillators are respectively given by

$$\Omega_i\,(t)^2 \equiv \omega\,(t)^2 \pm \frac{\lambda\,(t)}{\mu};\ \vec{\theta}_i\,(t) \equiv \frac{1}{\sqrt{2}}\left(\vec{f}_1\,(t) \pm\ \vec{f}_2\,(t)\right);\ i = 1, 2. \tag{32}$$

Furthermore, the path integral measure is invariant under this transformation. As a consequence, we are left to solve the problem of a free particle (coordinate $\vec{X}_3$), and two forced oscillators with time-dependent frequencies and forces [11]. Once the case of the oscillators is more involved, we review briefly a way to get the propagator for them. First of all, we perform a translation like

$$\vec{y}_i\ \equiv\ \vec{x}_i\ + \vec{\eta}_i\,, \tag{33}$$

and then impose the elimination of the linear terms, which make it necessary to restrict the arbitrary functions $\vec{\eta}_i$ through the equation

$$\frac{d^2\vec{\eta}_i}{dt^2} + \Omega_i\,(t)^2\ \vec{\eta}_i = -\frac{\vec{\theta}_i}{\mu}, \tag{34}$$

and this allow us to write the Lagrangians as

$$\mathcal{L}_j = \frac{\mu}{2}\left(\frac{d\vec{y}_j}{dt}\right)^2 - \frac{\mu}{2}\ \Omega_j\,(t)^2\ \vec{y}_j^{\,2}\ + \frac{d\,\vec{F}_j}{dt}\,;\ j = 1, 2 \tag{35}$$

with

$$\frac{d\,\vec{F}_j}{dt} \equiv \frac{1}{2}\mu\,\frac{d\vec{\eta}_j}{dt}\ \cdot (\vec{\eta}_j - 2\ \vec{y}_j) - \frac{1}{2}\int^t \vec{\theta}_j\,(\xi)\ \cdot \vec{\eta}_j\,(\xi)\ d\xi, \tag{36}$$

From the above, we conclude that the translation used has been able to reduce the problem to that of a harmonic oscillator with time-dependent frequency. At this point we can, for instance, map the problem into that of a free particle, what can be done by using a set of coordinate transformations, including a time substitution, introduced by Cheng many years ago [23], [9],

$$\vec{z}_i \equiv \vec{y}_i\ \dot{\alpha}_i^{1/2}\ \sec\left[\mu_i\,(t)\right],\ u_i \equiv \tan\left[\alpha_i\,(t)\right], \tag{37}$$

where $\alpha_i\,(t)$ and some auxiliary variables $s_i\,(t)$ should obey the equations

$$\ddot{s}_i + \Omega_i\,(t)^2\ s_i = \frac{1}{s_i^3},\ s_i^2\,\dot{\alpha}_i = 1. \tag{38}$$

After these transformations, we ends with the following Lagrangians

$$\mathcal{L}_j = \frac{\mu}{2}\left(\frac{d\vec{z}_j}{du_j}\right)^2 + \frac{d\,\vec{F}_j\left(\vec{y}_i,\vec{\eta}_i,t\right)}{dt} - \frac{dG_j\left(s_i,t\right)}{dt}, \tag{39}$$

with

$$G_j\left(s_i,t\right) \equiv \frac{1}{2}\mu\,\vec{y}_j^{\,2}\left\{\sin\left[2\,\mu_j\left(t\right)\right] - 2\frac{\ddot{s}_j}{s_j\,\dot{u}_j}\right\}. \tag{40}$$

Looking to the above equations, one can conclude that we have really mapped the original problem into that of free particles. Now, using the solution of the free particle oscillator [2] and the Van Vleck-Pauli formula [24], it can be shown after straightforward calculations that the i_{th} propagator becomes

$$\begin{aligned} K_{ia}\left(y^{''}_{ia},y'_{ia};\tau,0\right) \;&=\; \left(\frac{\sqrt{\dot{\alpha}'_{ia}\alpha^{"}_{ia}}}{2\,\pi\,i\,\sin\delta_{ia}}\right)^{\frac{1}{2}} \exp\left[\frac{i\,\mu}{2\,\hbar}\,\left(y^2_{ia}\frac{\dot{s}_{ia}}{s_{ia}} + \dot{\eta}_{ia}\left(\eta_{ia}-2\,y_{ia}\right)\right)_0^{\tau}\right]\times \\ &\times\,\exp\left\{\frac{i\,\mu}{2\,\hbar\,\sin\delta_{ia}}\left[\left(\,\alpha^{"}_{ia}\,\left(y_{ia}"\right)^2 + \,\dot{\alpha}'_{ia}\left(\,y'_{ia}\right)^2\right)\cos\delta_{ia} + \right.\right. \\ &\left.\left. -2\,\sqrt{\dot{\alpha}'_{ia}\alpha^{"}_{ia}}\;y'_{ia}\;y_{ia}"\right]\right\}\,\times \\ &\times\exp\left(-\frac{i}{2\,\hbar}\int_0^{\tau}\,\theta_{ia}\left(\xi\right)\eta_{ia}\left(\xi\right)\,d\xi\right), \end{aligned} \tag{41}$$

where it was define for the sake of compactification of the expressions that, for any function $g\left(t\right)$, one have that $g\left(\tau\right)\equiv g"$ and $g\left(0\right)\equiv g'$. Besides, in the above expression $\delta_{ia}\equiv\alpha^{"}_{ia}-\dot{\alpha}'_{ia}$.

$$\begin{aligned} K\left(\vec{Y}_1",\vec{Y}_1';\vec{Y}_2",\vec{Y}_2';\vec{Y}_3",\vec{Y}_3';\tau\right) \;&=\; \left(b^2\sqrt{\frac{m_1m_2M_1M_2}{m_3m_{12}^2M}}\right)^3\left(\frac{M}{2\,\pi\,i\,\tau}\right)^{\frac{3}{2}}\times \\ &\times\exp\left\{\frac{i\,M}{2\,\hbar\,\tau}\left(\vec{X}_3"-\vec{X}_3'\right)^2\right\} \\ &\times\Pi_{i=1}^2\Pi_{a=1}^3\,K_{ia}\left(\vec{y}_{ia}",\vec{y}_{ia}';\tau\right). \end{aligned} \tag{42}$$

Let us now recover the wave functions through the use of the decomposition of the propagator in terms of the wave functions, as it is given from (20), and again using the Mehler's formula (21), with the necessary identifications

$a_i = \sqrt{\frac{\mu \acute{\dot{\alpha}_i}}{\hbar}}\, \acute{y_i}$, $b_i = \sqrt{\frac{\mu \acute{\dot{\alpha}_i}}{\hbar}}\, \acute{y_i}$ and $c_i = \exp\left(-i\,\delta_i\right)$, we obtain for the discrete part of the wave functions

$$\Psi\left(\vec{X}_1, \vec{X}_2, \vec{X}_3\right) = \Pi_{a=1}^{3} \psi_{n_{1a},n_{2a}}\left(X_{1a}, X_{2a}\right) \exp\left[i\,\vec{k}\cdot\,\vec{X}_3\right], \tag{43}$$

with

$$\begin{aligned}
\psi_{n_{1a},n_{2a}}\left(X_{1a}, X_{2a}\right) \;=\; & \left[\frac{1}{2^{n_{1a}+n_{2a}}\left(n_{1a}!\,n_{2a}!\right)}\left(\frac{\mu^2\,\dot{\alpha}_1\,\dot{\alpha}_2}{\left(\pi\,\hbar\right)^2}\right)\right]^{\frac{1}{2}} \\
& \times \exp\left\{\frac{i\,\mu}{4\,\hbar}\left[\left(X_{1a} - X_{2a} + \sqrt{2}\,\eta_{1a}\right)^2\left(\frac{\dot{s}_1}{s_1} - \dot{\alpha}_1\right) + \right.\right. \\
& \left.\left. -2\dot{\eta}_{1a}\left(\eta_{1a} + \sqrt{2}\left(X_{1a} - X_{2a}\right)\right)\right]\right\} \\
& \times \exp\left\{\frac{i\,\mu}{4\,\hbar}\left[\left(X_{1a} + X_{2a} + \sqrt{2}\,\eta_{2a}\right)^2\left(\frac{\dot{s}_2}{s_2} - \dot{\alpha}_2\right) + \right.\right. \\
& \left.\left. -2\dot{\eta}_{2a}\left(\eta_{2a} + \sqrt{2}\left(X_{1a} + X_{2a}\right)\right)\right]\right\} \\
& \times H_{n_{a1}}\left(\left[\frac{\mu\,\dot{\alpha}_1}{2\,\hbar}\right]^{\frac{1}{2}}\left(X_{1a} - X_{2a} + \sqrt{2}\,\eta_{1a}\right)\right) \qquad (44) \\
& \times H_{n_{a2}}\left(\left[\frac{\mu\,\dot{\alpha}_2}{2\,\hbar}\right]^{\frac{1}{2}}\left(X_{1a} + X_{2a} + \sqrt{2}\,\eta_{2a}\right)\right) \\
& \times \exp\left[-\frac{i}{2\,\hbar}\int^{\tau}\left(\theta_{1a}\left(\xi\right)\eta_{1a}\left(\xi\right) + \theta_{2a}\left(\xi\right)\eta_{2a}\left(\xi\right)\right)\,d\xi\right] \\
& \times \exp\left\{-i\left[\left(n_{1a} + \frac{1}{2}\right)\alpha_1\right] + \left(n_{2a} + \frac{1}{2}\right)\alpha_2\right\}.
\end{aligned}$$

Finally we would like to remark that some other systems which are somewhat more general than that appeared until now in the literature could be included in this analysis. This is the case, for instance, of a three-dimensional generalization of the potential considered in the interesting work of Chouchaoui [1], which would be represented by the following Lagrangian

$$\begin{aligned}
\mathcal{L} \;=\; & \sum_{j=1}^{3}\frac{m_j}{2}\left(\frac{d\vec{r}_j}{dt}\right)^2 - \frac{1}{2}\left[K_{21}\,\left(\vec{r}_2 - \vec{r}_1\right)^2 + K_{31}\,\left(\vec{r}_3 - \vec{r}_1\right)^2 + K_{32}\,\left(\vec{r}_3 - \vec{r}_2\right)^2 + \right. \\
& + \sigma_1\left(\vec{r}_2 - \vec{r}_1\right)\cdot\left(\vec{r}_3 - \vec{r}_1\right) + \sigma_2\left(\vec{r}_2 - \vec{r}_1\right)\cdot\left(\vec{r}_3 - \vec{r}_2\right) + \qquad (45) \\
& \left. +\sigma_3\left(\vec{r}_3 - \vec{r}_1\right)\cdot\left(\vec{r}_3 - \vec{r}_2\right) + \frac{g_1}{\left(\vec{r}_2 - \vec{r}_1\right)^2}\right],
\end{aligned}$$

and with some restrictions over the potential parameters. Furthermore, we could also include in the list of the systems which can be decoupled through the use of the transformations above described, some cases where particular three-body interactions could added to the general quadratic coordinate one. More precisely, one could accrescent something like

$$\Delta V = \frac{g_1}{(\vec{r}_2 - \vec{r}_1)} + \frac{g_2}{(m_{12}\vec{r}_3 - (m_1\vec{r}_1 + m_2\vec{r}_2))^2} + \frac{g_3}{(m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3)^2},$$

to the interacting potential. But, once again, restrictions over the parameters of the potential should be imposed in order to keep the exactness of the propagator and wave functions solutions. Moreover, the case of the general quadratic spatial interactions, which we considered in this work, has the advantage of presenting covariant interactions between the three particles involved. For this reason, we prefer do not treat these cases here.

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